**Cost Structures**

Suppose we have a minimization model that includes a commodity that is priced to discourage consumption (e.g., first 50 units cost $20/unit, next 40 units cost $30/unit).

Let x = # of units consumed

Let f(x) = cost

f(x) = 20x 0 <= x <= 50

30x – 500 50 <= x <= 90

Note, the reason it is 30x – 500; is 50\*20 + 30(x – 50).

(ie. Regular time, overtime production)

Let x = x1 + x2. Where x1 = # of units @ $20; and x2 = # of units @ $30

Objective Term: 20x1 + 30x2

Constraints: x1 <= 50

x1 + x2 <= 90

x = x1 + x2

*This is an example of piecewise linear within the topics of optimization.*

*Note, another example of this is regular time and overtime.*

**NOW** … Suppose instead the cost is discounted (e.g., the first 50 units cost $30/unit, next 40 cost $20/unit) to model this, we need to use the concept of convex combination. – encourage consumption

Note, a convex combination is given 2 points (x1 and x2) the convex combination of x1 and x2 is:

λx1 + (1 - λ) x2 parametrically defines the line; where 0 <= λ <= 1

Note, could also be written as λ1x1 + λ2x2 where λ1 + λ2 = 1 and λi >= 0

The first 50 units cost $30/unit, next 40 cost $20/unit

Let x = total # of units; Let f(x) be cost function.

f(x) = 30x 0 <= x <= 50

1500 + 20(x – 50) 50 <= x <= 90

Note, the second element can be rephrased to: 500 + 20x

* This situation is harder to model since we have to consume the first 30 first.

Breakpoints:

t1 = 0

t2 = 50

t3 = 90 (or some other upper limit)

If 0 <= x <= 50; we want to use Segment 1; If 50 <= x <= 90; we want to use Segment 2

Let y1 = 1 if segment 1 is used; 0 otherwise; Let y2 = 1 if segment 2 is used; 0 otherwise

Thus, y1 + y2 = 1

(one segment at a time)

If y1 = 1 then x can be written as a convex combination of t1 and t2:

x = t1(λ1) + t2(λ2); where λ1 + λ2 = 1; λ >= 0 for all i

x = 0(λ1) + 50(λ2)

Now plug-in x into f(x):

f(x) = 0(λ1) + 1500(λ2)

If y2 = 1 then x can be written as a convex combination of t2 and t3:

x = t2(λ2) + t3(λ3); where λ2 + λ3 = 1; λ >= 0 for all i

x = 50(λ2) + 90(λ3) (how far from 50 and 90)

Now plug-in x into f(x):

f(x) = 1500(λ2) + 2300(λ3)

Goal: Want Y’s to control λ’s

x = t1 λ1 + t2 λ2 + t3 λ3

f(x) = f(t1) λ1 + f(t2) λ2 + f(t3) λ3 (Nonlinear, but being written as linear by transformation)

Note, for our particular problem: x = 0(λ1) + 50(λ2) + 90(λ3), cost = f(x) = 0(λ1) + 1500(λ2) + 2300(λ3)

λ1 + λ2 + λ3 = 1

λ1 <= Y1 (if Y1 is 0, λ1 has to be 0)

λ2 <= Y1 + Y2

λ3 <= Y2

Y1 + Y2 = 1

Y’s binary

λi >= 0 for all i

Example:

A company manufactures two products A and B, which are produced using two raw materials. Currently both products are in high demand and it is anticipated that all that is manufactured can be sold. The selling price for one unit of Product A is $3.20 and the selling price for Product B is $2.70. Product A requires **4.2 pounds of Raw Material 1**, **and 1.7 pounds of Raw Material 2**, while the corresponding figures for Product B are **3.9 pounds and 1.1 pounds**. Raw Materials 1 and 2 are purchased from a local vendor and the purchase prices are determined according to the following schedule.

Raw Material 1 Raw Material 2

First 500 pounds $0.30/lb First 200 pounds $0.40/lb

Next 1000 pounds $0.25/lb Next 600 pounds $0.35/lb

Next 1000 pounds $0.20/lb

Maximum Available: 2500 pounds Maximum Available: 800 pounds

Prior to sale, Products A and B must also undergo final inspection and packaging. Final inspection and packaging of each unit of Product A requires 1.2 hours, whereas Product B requires 0.9 hours per unit. If there are 480 hours available for final inspection and packaging, formulate an IP problem for determining the product mix which maximizes profit.

Initial Problem Formulation (without cost break points):

A: # of product A

B: # of product B

R1: # of Raw Material 1

R2: # of Raw Material 2

f1(R1): cost of Raw Material 1

f2(R2): cost of Raw Material 2

Maximize Z = 3.2A + 2.7B – f1(R1) – f2(R2)

ST

R1 = 4.2A + 3.9B

R2 = 1.7A + 1.1B

1.2A + 0.9B <= 480

R1 <= 2500

R2 <= 800

A, B, R1, R2 >= 0

NOW … let’s look at the cost functions for R1 and R2.

Breakpoints: R1

ti: 0, 500, 1500, 2500; f(ti): 0, 150, 400, 600

f1(R1): 0.3R1 0 <= R1 <= 500

150 + 0.25(R1 – 500) 500 <= R1 <= 1500

Update: *25 + 0.25R1*

400 + 0.2(R1 – 1500) 1500 <= R1 <= 2500

Update: *100 + 0.2R1*

Breakpoints: R2

ti: 0, 200, 800; f(ti): 0, 80, 290

f2(R2): 0.4R2 0 <= R2 <= 200

80 + 0.35(R2 – 200) 200 <= R2 <= 800

Update: 10 + 0.35R2

Now … reset the formulation with the cost functions and convex combinations.

Using λ and Y for R1 and α and W for R2.

Max Z = 3.2A + 2.7B – [0(λ1) + 150(λ2) + 400(λ3) + 600(λ4)] – [0(α1) + 80(α2) + 290(α3)]

ST

R1 = 4.2A + 3.9B

R2 = 1.7A + 1.1B

1.2A + 0.9B <= 480

R1 <= 2500

R2 <= 800

R1 = 0(λ1) + 500(λ2) + 1500(λ3) + 2500(λ4)

R2 = 0(α1) + 200(α2) + 800(α3)

λ1 + λ2 + λ3 + λ4 = 1

α1 + α2 + α3 = 1

Y1 + Y2 + Y3 = 1

λ1 <= Y1

λ2 <= Y1 + Y2

λ3 <= Y2 + Y3

λ4 <= Y3

λi >= 0 for all i

Yi = binary for all i

W1 + W2 = 1

α1 <= W1

α2 <= W1 + W2

α3 <= W2

αi >= 0 for all i

Wi = binary for all i